UNIFORM FIELD FROM DISTRIBUTION OF CURRENTS ON AN ELLIPSE

S. C. Snowdon

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Purpose

To determine the distribution of currents on the surface of an ellipse that will produce a uniform field within the ellipse. Extention to a gradient field is also given.

Reference

1. W. R. Smythe, Static and Dynamic Electricity, McGraw-Hill Book Co., Inc., New York (1950).

Coordinate System

Conformal transformations[1] suggest that the variables (u,v) are useful when dealing with problems having an elliptical boundary.

$$x = a \sin u \, Ch \, v \qquad y = a \cos u \, Sh \, v \qquad (1)$$

The factor for displacement is

$$h_x = h_y \equiv h = a \sqrt{Sh^2 v + \cos^2 u} \qquad (2)$$

Potentials (Uniform Field)

Since one desires a uniform inside the ellipse one chooses

$$\phi_i = -B_0 y = -B_0 a \cos u \, Sh \, v \qquad (3)$$

The transformation of Eq. (1), being conformal gives for the Laplace equation

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} = 0. \qquad (4)$$

A solution appropriate to the external region is

$$\phi_0 = A \, a \cos u \, e^{-v} \qquad (5)$$

Boundary Conditions

The normal component of the flux density is continuous.
This gives

$$\left(\frac{\partial \phi_i}{\partial v}\right)_{v = v_0} = \left(\frac{\partial \phi_0}{\partial v}\right)_{v = v_0} , \tag{6}$$

where from Eq. (1) $v = v_0$ is seen to generate an ellipse. Thus

$$A = B_0 e^{v_0} \, Ch \, v_0. \tag{7}$$

Applying the Ampere circuital law to a small region
spanning $v = v_0$ and assuming a surface current of density σ exists
on the surface gives

$$\left(H_{u0} - H_{ui}\right)_{v = v_0} h \, \Delta \, u = 4 \, \pi \, \sigma \, h \, \Delta \, u. \text{ (emu)} \tag{8}$$

or

$$\sigma = \frac{1}{4\pi}\left(\frac{\partial \phi_i}{h\partial u} - \frac{\partial \phi_0}{h\partial u}\right)_{v = v_0} \tag{9}$$

or

$$\sigma = \frac{B_0}{4\pi} \cdot e^{v_0} \cdot \frac{Sin \, u}{\sqrt{Sh^2 v_0 + cos^2 u}} \quad \text{(emu)} \tag{10}$$

Potentials (Gradient Field)

One desires to have inside the ellipse

$$\phi_i = -B'_0 x \, y = -\frac{a^2 B'_0}{4} \sin 2 \, u \, Sh \, 2 \, v \tag{11}$$

Outside the ellipse there are no sources. Hence

$$\phi_0 = 1/4 \, a^2 \, A \sin 2 \, u \, e^{-2v} \tag{12}$$

Matching the normal fields on the elliptical boundary gives

$$A = B'_0 e^{2v_0} \text{Ch } 2 v_0. \tag{13}$$

The surface current density is then determined from the discontinuity in the tangential component of the field.

$$\sigma = - \frac{aB'_0}{8\pi} e^{2v_0} \frac{\text{Cos } 2 u}{\sqrt{\text{Sh}^2 v_0 + \cos^2 u}} \qquad \text{(emu)} \tag{14}$$